Probability Distribution and Noise Factor of Solid State Photomultiplier Signals with Cross-Talk and Afterpulsing

S. Vinogradov, T. Vinogradova, V. Shubin, D. Shushakov, and K. Sitarsky

Abstract—**Operating principles of Solid State Photomultipliers are based on Geiger mode avalanche breakdown limited by strong negative feedback. This operating mode provides both high gain and ultra-low excess noise of avalanche multiplication resulting in ability to detect single photons. On the other hand high gain is accompanied with cross-talk and afterpulsing processes changing the probability distribution function of output signals and arising specific excess noise. There are many reports regarding its influence on signal distribution and noise, and we suggest that some analytical expressions would be useful.**

We therefore are presenting a simple model of probability distribution of output signals in the presence of cross-talk and afterpulsing (namely compound Poisson distribution). The model results allow us to calculate excess noise factor of cross-talk and afterpulsing as well as to find simple way to measure its probabilities.

Index Terms — Afterpulsing, Crosstalk, Noise Factor, Probability Distribution, Solid State Photomultipliers.

I. Introduction

Use of overcritical avalanche process with negative feedback (limited Geiger mode) for proportional detection of an extra-low light signal was first proposed more than 10 years ago [1]-[5]. Limited Geiger mode avalanche multiplication is characterized by high gain (10^4 – 10^6) and ultra-low excess noise factor (1.01 – 1.05) due to negative feedback that is responsible for the quenching of breakdown with efficient suppression of output charge fluctuations. Photodetectors utilizing this mode are known now as Solid State Photomultiplier (SSPM), Silicon Photomultiplier, Geiger mode APD matrix, Multi-Pixel Photon Counter, and a few more names. Let us therefore use the collective name SSPM for this generation of photodetectors. To perform proportional detection of light pulses, these devices are typically designed as matrix of APD pixels operating in limited Geiger mode with common output. Multi-pixel architecture with low noise high gain multiplication of each pixel is the base for the SSPM advantage in a few photon pulse detection, but on the other hand it is the source of considerable cross-talk and afterpulsing processes sacrificing photodetection quality.

Manuscript received October 17, 2009.

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Every specific application of SSPM requires optimal selection of operating voltage to balance the signal performance (gain, photon detection efficiency, timing resolution, and so on) with noise (dark count rate, cross-talk, and afterpulsing).

The goal of this paper is to find the analytical expressions clarifying the influence of cross-talk and afterpulsing on photodetection characteristics that are useful for such optimization.

Physical models of these processes are widely reported and discussed [6]-[10].

Cross-talk in SSPM is the immediate firing of a neighboring pixel by the initially fired one. In the most practical cases the secondary pulse from the neighboring pixel is produced simultaneously with the initial pulse. As a result the primary pulse from one pixel (true event) is duplicated by the secondary one from another pixel (false event) and a double amplitude pulse is observed in the output. The afterpulse is a delayed secondary pulse firing of the same pixel. In some cases we observed that cross-talk pulses may be delayed from the primary pulses and appear like afterpulses [11]. In practice if we detect total charge or total number of events in a time gate we cannot distinguish the cross-talk and afterpulsing origin of pulses, so let us call false events of any kind as duplicated events and its generation process as duplication.

Let us consider a typical experimental result of a few photon short pulse detection as illustrated in Fig. 1 from [12]. This histogram shows a distribution of output charge, which reflects the probabilities to detect photoresponse pulses equal to 0, 1, 2, and more fired pixels (referred to as "photoelectrons"). The histogram peaks are very narrow due to low excess noise of charge multiplication, so the probability distribution function may be easily reconstructed from the histogram. If the number of duplicated events is negligible the probability distribution should follow Poisson law. However in a lot of experiments, when cross-talk and afterpulsing are considerable, show deviations from the Poisson law.

There were some attempts to find relationships between

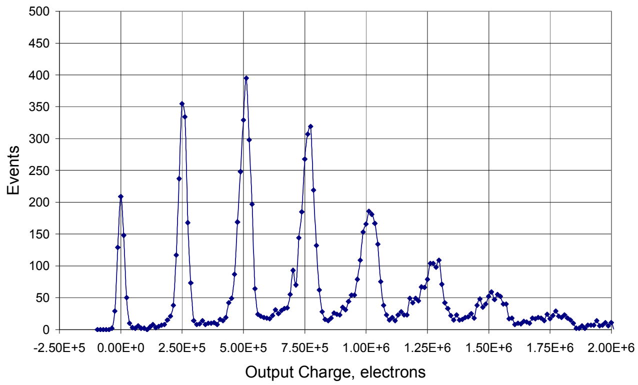

Fig. 1. Output charge distribution histogram: detection of 60 ps 442 nm laser pulses of 13 photon/pulse in 3 ns gate by 1 mm DAPD SSPM.

such histograms (experimental probability distribution function) and the characteristics of duplication processes, but we assume that they did not result in a complete solution with universal analytical expressions.

For example, Fourier transform expression was derived in [13] to reconstruct a histogram by using some measured and fitting parameters.

Iterative approximation of a probability distribution function was proposed in [14], but it may be applied only for a small number of events and low probability of a duplicated event (less then 10% by our estimation).

A number of specific analyses of afterpulsing in a vacuum PMT were carried out taking into account very low probability of afterpulses [15]-[18].

Monte Carlo simulations of stochastic processes of photodetection including cross-talk and afterpulsing became popular in last years [19], [20], but they were unable to derive analytical model.

II. Probability Distribution Model

A. Assumptions

We assume that any pulse produced by any pixel is identical regardless of primary or secondary origin of the pulse (so-called Single Electron Response). The number of pulses detected in a time gate is the sum of some primary and some secondary pulses.

We assume that the following conditions are satisfied:

- Random variable – number of primary pulses produced by incident photons (or dark electrons) – belongs to Poisson distribution;
- Every primary event may produce an infinite chain of secondary events with the same probability p for each secondary event; hence random variable – number of secondary pulses – belongs to a geometric distribution;
- Geometric distribution law corresponds to random processes caused by both cross-talk and afterpulsing;
- All produced events are measured including all events in the tails.

Thus random variable X – total number of detected pulses – is expressed as a compound Poisson distribution:

$$X = \sum_{i=1}^{N} (1 + G_i) \tag{1}$$

where

G_i - independent variables of geometric distribution with parameter p,

$$P(G_i = k) = p^k (1-p) \quad for \;\; k = 0, 1, 2, 3\dots,$$

N - random variable, independent from G_i and belonging to Poisson distribution with parameter (mean value) L.

B. Characteristics of Compound Poisson Distribution

We are interested in the probability distribution function of X

$$f_k(p, L) = P(X = k) \quad for \;\; k = 0, 1, 2, 3\dots, \tag{2}$$

as well as in mean EX and variance $Var(X)$.

We use the probability generating function to calculate analytical expressions of our interest from (1). In correspondence with [21], the generating function for discrete random variable φ is determined as

$$\Phi(s) = \sum_{i=0}^{\infty} P(\varphi = i) \times s^i . \tag{3}$$

The generating function allows us to find probability distribution function, mean and variance of random variable φ:

$$P(\varphi = k) = \frac{1}{k!} \times \frac{d^k}{ds^k} \Phi(0)$$

$$E\varphi = \Phi'(1) \tag{4}$$

$$Var(\varphi) = \Phi''(1) + \Phi'(1) - [\Phi'(1)]^2$$

Turning to our specific case let us define

$F(s, p, L)$ - generating function for X,

$g(s, p)$ - generating function for $(1 + G_i)$. (5)

As known, generating function for X belonging to compound Poisson distribution is expressed as:

$$F(s, p, L) = \exp(-L + L \times g(s, p)) \tag{6}$$

Using generating function definition we get

$$g(s, p) = \sum_{i=0}^{\infty} P\{(1+G) = i\} \times s^i$$

$$= \sum_{i=1}^{\infty} P\{G = i-1\} \times s^i \tag{7}$$

$$= \sum_{i=1}^{\infty} p^{i-1} \times (1-p) \times s^i = \frac{(1-p) \times s}{1 - p \times s}$$

Thus,

$$F(s,p,L) = \exp(L\frac{s-1}{1-p\times s}) \quad (8)$$

Using (4) and (8) we can derive the probabilities to detect totally 0, 1, 2 pulses, and more (see Appendix for details):

$$f_0(p,L) = \exp(-L),$$
$$f_1(p,L) = \exp(-L)\times L\times(1-p),$$
$$f_2(p,L) = \exp(-L)\times\left[L\times(1-p)\times p + \frac{L^2\times(1-p)^2}{2}\right] \quad (9)$$

We also can derive mean and variance:

$$EX = \frac{L}{(1-p)}, \quad Var(X) = \frac{L\times(1+p)}{(1-p)^2}. \quad (10)$$

Let us note that we may rewrite (10) in term of mean number of duplicated pulses in a chain produced by single primary pulse or "coefficient of duplication" K_{dup}. It is equal to mean of geometric distribution:

$$K_{dup} = \frac{p}{1-p},$$
$$EX = L\times(1+K_{dup}), \quad (11)$$
$$Var(X) = L\times(1+K_{dup})\times(1+2\times K_{dup})$$

So, the mean of compound Poisson distribution is a product of the Poisson distribution mean L and the geometric distribution mean including one primary event $1+K_{dup}$. And variance of the compound Poisson distribution exceeds its mean value by factor of $1+2\times K_{dup}$ (so-called Fano factor).

This fact results in an increase of noise to signal ratio for photodetection in the presence of duplication processes.

III. Probability Distribution Features

Examples of a compound Poisson distribution function (9) are presented on Fig. 2 and Fig 3.

The plots allow us to observe how the increase of secondary event probability results in the increase of mean signal value and the increase of distribution width (10).

Our analysis also shows some other features of compound Poisson distribution:

- if the mean is high then the compound distribution approaches to Gauss distribution with the same mean and variance;
- the higher probability p the higher mean when this convergence appears;
- the higher probability p the more asymmetric shape of compound Poisson distribution.

Expressions (9) allow one to easily evaluate compound Poisson distribution parameters L and p from first (f_0) and second (f_1) peaks of experimental distribution histogram:

$$L = -\ln(f_0) \quad (12)$$

$$p = 1 + \frac{f_1}{f_0\times\ln(f_0)} \quad (13)$$

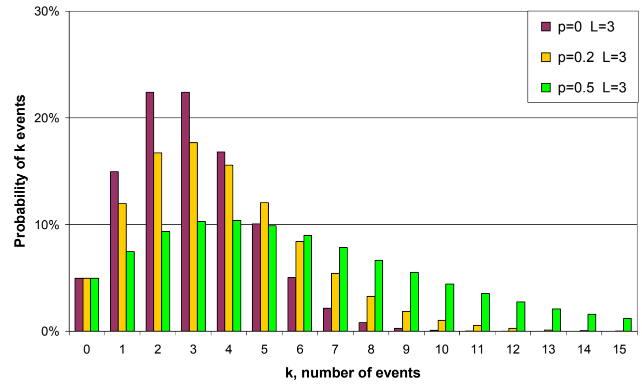

Fig. 2. Compound Poisson distribution function for L = 3, p = 0, 0.2, and 0.5; (p = 0 for pure Poisson law).

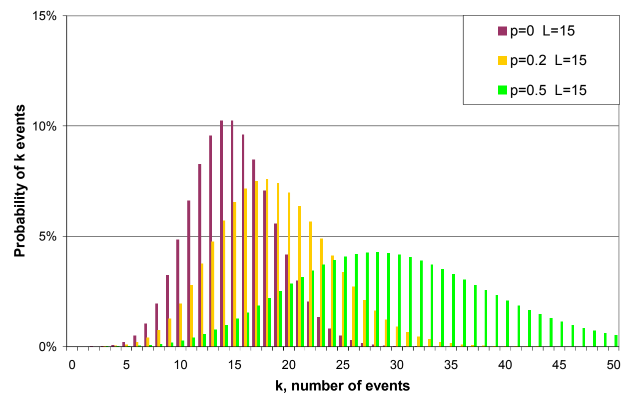

Fig. 3. Compound Poisson distribution function for L = 15, p = 0, 0.2, and 0.5; (p = 0 for pure Poisson law).

Equation (12) is widely used in evaluation of photon detection efficiency (PDE), when the mean number of photons per detected pulse Nph is known:

$$L = Nph\times PDE \quad (14)$$

Equation (13) reflects the fact that probability to detect one pulse f_1 is a product of the probability to detect one primary Poisson pulse and none of secondary pulses (9). It may be noted that duplication probability p may be determined by known f_0 and f_1 values (even if L is unknown) from histogram of any kind, for example, for dark events only.

Let us apply this approach to evaluation of the experimental result presented on Fig.1. Experimental data on Fig. 4 is number of events in corresponding peak of the histogram normalized to the total number of events in the histogram. Model data is a compound Poisson distribution based on values L and p evaluated using (12) and (13). Poisson data is a pure Poisson distribution with the same L (actually, it is equal to compound Poisson distribution with p =0).

In our opinion the accuracy of evaluation of L and p may be improved by using expressions (9) for higher order peaks of the histogram.

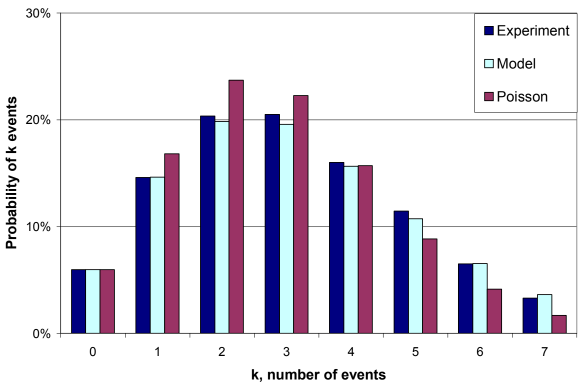

Fig. 3. Experimental data from Fig. 1 in comparison with the model for mean L = 2.8 (PDE=22% @Nph=13), p = 13% and with pure Poisson law.

IV. Excess Noise Factor of Secondary Events

It is well-known that linear mode APD is characterized by high noise of multiplication due to stochastic fluctuation of multiplied charge carriers at output. The multiplication mechanism of vacuum PMT has very limited but not negligible noise. SSPM multiplication noise is almost negligible in practice, but as we can see from (11) duplications cause the degradation of signal to noise ratio (SNR). Let us try to present a comparative evaluation of these devices in the sense of its relative noises, namely by using excess noise factor criteria.

Excess noise factor (ENF) for amplification of a noisy signal is defined as the squared SNR degradation rate from input to output of the detector:

$$ENF = \left(\frac{SNR\_in}{SNR\_out} \right)^2$$
$$SNR = \frac{EX}{\sqrt{Var(X)}} \qquad (15)$$

Assuming a Poisson distribution of the input signal with the corresponding primary events and using (10) for the output signal we get:

$$SNR_{in} = \sqrt{L}$$
$$SNR_{out} = \frac{\sqrt{L}}{\sqrt{1+p}} \qquad (16)$$

Thus we find a very simple expression for ENF caused by stochastic noise of secondary events:

$$ENF\_dup = 1 + p \qquad (17)$$

Obviously, (17) means that value of ENF caused by cross-talk and afterpulsing is less then 2. Thus SSPM (typical ENF of limited Geiger mode avalanche multiplication is 1.01 – 1.05) with cross-talk and afterpulsing is less noisy then conventional APD (typical ENF of avalanche multiplication in linear mode is 3 - 10), and it may be less noisy then PMT (typical ENF of dynode cascade multiplication is 1.2 – 1.3) only if the probability of duplication events p is less then 20 - 30%.

Considering ENF of photodetector by applying the definition (15) to incident photon detection process with probability PDE to detect single photon and probability p to detect single duplication and using (14) and (16) we can express the total ENF_total as a product of excess noise factor of photon detection (equal to $1/PDE$) and excess noise factor of duplication (17):

$$ENF\_total = \frac{1+p}{PDE} \qquad (18)$$

Actually (18) shows one of the most important criteria of operating voltage optimization, namely at the minimum of ENF which exists due to commonly observed [22] sub-linear dependence of PDE on voltage and super-linear dependence of p .

V. Conclusion

The proposed model may be applicable in a few ways. In our opinion it is useful first for analysis of the output signal probability distributions affected by cross-talk when the output signal includes all events from the cross-talk chain. It happens in measure of the output signal pulse amplitudes in the dark or resulting from short laser pulses, as well as in measure of the output charge in a narrow time gate separating cross-talk from afterpulsing. Analysis of afterpulsing may be carried out in counting mode when the time gate for a single measure is wider then the typical duration of afterpulsing chains and cross-talk events are excluded from the output counts (amplitudes of pulses are not distinguished).

We must note that the applicability of the model to real SSPM is limited by the linear dynamic range of the output signal and the reasonable values of duplication probability. It means that the geometric distribution law of duplication process should not be affected by SSPM saturation: for example, the number of triggered pixels should be considerably lower then the total number of pixels. In the opposite case if the output signal is close to saturation then the length of duplication chains should be considered as finite and the probability of duplication becomes variable, strongly dependent on number of non-triggered pixels. This case requires essential modification of the model.

VI. Appendix

Analyzing model results derived from generating function approach (4) and (8) we found another expression to calculate compound Poisson distribution function:

$$f_k(p,L) = \frac{\exp(-L)\cdot\sum_{i=0}^{k} B_{i,k}\cdot[L(1-p)]^i \cdot p^{k-i}}{k!}$$

where (19)

$$B_{i,k} = \begin{cases} 1 \quad if \quad i=0 \; and \; k=0 \\ 0 \quad if \quad i=0 \; and \; k>0 \\ \frac{k!\cdot(k-1)!}{i!\cdot(i-1)!\cdot(k-i)!} \quad otherwise \end{cases}$$

It may be more convenient in practical use.

We verify correctness of this expression by using symbolic calculations and comparing it with generating function based approach for $k = 0\ldots34$. We are sure it is correct for any k.

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